Laplace Equation in Spherical Coordinates

Laplace equation is given by

$$\nabla^2 V = 0$$

In spherical coordinates (r, θ, ϕ), Laplace equation is written as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rV) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \quad \text{--- (1)}$$

We use separation of variables to find the solution of above equation. We assume that potential is written in the product form as

$$V(r,\theta,\phi) = \frac{R(r)}{r} P(\theta) Q(\phi) \quad \text{--- (2)}$$

To simplify the algebra we have included an extra factor '$\frac{1}{r}$'. Next, using eq. (2) in equation (1)

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left[r \cdot \frac{R(r)}{r} P(\theta) Q(\phi)\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial}{\partial \theta}\left(\frac{R(r)}{r} P(\theta) Q(\phi)\right)\right]$$
$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}\left[\frac{R(r)}{r} P(\theta) Q(\phi)\right] = 0$$

or
$$P(\theta) Q(\phi) \frac{1}{r}\frac{\partial^2 R(r)}{\partial r^2} + \frac{R(r)}{r} Q(\phi) \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial P(\theta)}{\partial \theta}\right)$$
$$+ \frac{R(r)}{r} P(\theta) \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 Q(\phi)}{\partial \phi^2} = 0$$

Next, we multiply above expression by $\frac{r^3 \sin\theta}{R(r)P(\theta)Q(\phi)}$

we obtain,

$$\frac{r^2\sin^2\theta}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{\sin\theta}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right) + \frac{1}{Q(\phi)}\frac{d^2Q(\phi)}{d\phi^2} = 0 \quad \text{--- (3)}$$

The third term is independent of r and θ, thus it must be equal to a constant.

$$\frac{r^2\sin^2\theta}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{\sin\theta}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right) - m^2 = 0 \quad \text{--- (4)}$$

and

$$\frac{1}{Q(\phi)}\frac{d^2Q(\phi)}{d\phi^2} = -m^2 \quad \text{--- (5)}$$

Eq. (5) can be solved.

$$\frac{d^2Q(\phi)}{d\phi^2} = -m^2Q(\phi)$$

$$\left(\frac{d^2}{d\phi^2} + m^2\right)Q(\phi) = 0$$

Solution is

$$Q(\phi) = A_m e^{im\phi} + B_m e^{-im\phi}, \quad m \neq 0$$

for $m=0$, soln $\rightarrow Q(\phi) = A_0 + B_0\phi$ --- (6)

Again taking Eq. (4)

$$\frac{r^2\sin^2\theta}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{\sin\theta}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right) - m^2 = 0$$

dividing by $\sin^2\theta$,

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{1}{\sin\theta\, P(\theta)}\frac{d}{d\theta}\left(\sin\theta \frac{dP(\theta)}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0 \quad (7)$$

The last term and first term are independent and therefore we can write

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2} - l(l+1) = 0 \quad (8)$$

where $$\frac{1}{\sin\theta\, P(\theta)}\frac{d}{d\theta}\left(\sin\theta \frac{dP(\theta)}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -l(l+1) \quad (9)$$

Now eqs. (8) & (9) can be written as

$$\frac{d^2R(r)}{dr^2} = l(l+1)\frac{R(r)}{r^2}$$

and $$\frac{d}{d\theta}\left[\sin\theta \frac{dP(\theta)}{d\theta}\right] + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\sin\theta\, P(\theta) = 0$$

For the last equation we take $x = \cos\theta$, which gives $\frac{d}{d\theta} = -\sqrt{1-x^2}\frac{d}{dx}$. Therefore, we write

$$\frac{d^2R(r)}{dr^2} = l(l+1)\frac{R(r)}{r^2} \quad (10)$$

and $$\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P(x) = 0 \quad (11)$$

Let us try solution of the form $R(r) = r^\alpha$ of eqn. (10)

$$\frac{d^2(r^\alpha)}{dr^2} = l(l+1)\frac{r^\alpha}{r^2}$$

$$\alpha(\alpha-1)r^{\alpha-2} = l(l+1)r^{\alpha-2}$$

or $$\left[\alpha^2 - \alpha - l(l+1)\right] r^{\alpha-2} = 0$$

or $$\alpha^2 - \alpha - l(l+1) = 0$$

which gives $\alpha = l+1$, $\alpha = -l$

Therefore the general solution is given by

$$R(r) = A_l r^{l+1} + B_l r^{-l}$$

Now we write the general solution of eqn. (1)

(see eqn. (2)).

$$\boxed{V(r,\theta,\phi) = \sum_m \sum_l \left(A_l r^l + B_l r^{-l-1}\right)\left(A_m e^{im\phi} + B_m e^{-im\phi}\right) P_l^m(\cos\theta)}$$ —— (12)

where $P_l^m(\cos\theta)$ is the solution of the equation having θ dependence, the equation (11).

For the special case $m=0$, we solve equation (11) for $P(x)$.

The eqn. (11) for $m=0$ is

$$\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right] + l(l+1)P(x) = 0 \quad \text{—— (13)}$$

Let us try a power series solution of the form

$$P(x) = \sum_{j=0}^{\infty} a_j x^{j+\alpha} \quad \text{(14)}$$

using (14) in (13),

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\sum_j a_j x^{j+\alpha}\right] + l(l+1)\sum_j a_j x^{j+\alpha} = 0$$

$$\frac{d}{dx}\left[(1-x^2)\sum_j a_j (j+\alpha) x^{j+\alpha-1}\right] + l(l+1)\sum_j a_j x^{j+\alpha} = 0$$

$$\sum_j \left\{(j+\alpha)(j+\alpha-1) a_j x^{j+\alpha-2} - (j+\alpha)(j+\alpha+1) a_j x^{j+\alpha}\right\} + l(l+1)\sum_j a_j x^{j+\alpha} = 0$$

$$\sum_{j=0}^{\infty} (j+\alpha)(j+\alpha-1) a_j x^{j+\alpha-2} - \sum_{j=0}^{\infty}\left[(j+\alpha)(j+\alpha+1) - l(l+1)\right] a_j x^{j+\alpha} = 0$$

Combining the same powers in one sum

$$\alpha(\alpha-1) a_0 x^{\alpha-2} + (1+\alpha)\alpha a_1 x^{\alpha-1} + \sum_{j=0}^{\infty}\left[(j+\alpha+2)(j+\alpha+1) a_{j+2} - \{(j+\alpha)(j+\alpha+1) - l(l+1)\} a_j\right] x^{j+\alpha} = 0$$

above expressions is true for all x, therefore, coefficients of each power is equal to zero

$$\alpha(\alpha+1) a_0 = 0 \quad \text{--- (15)}$$

$$(1+\alpha)\alpha a_1 = 0 \quad \text{--- (16)}$$

$$(j+\alpha+2)(j+\alpha+1) a_{j+2} - [(j+\alpha)(j+\alpha+1) - l(l+1)] a_j = 0$$

$$\downarrow$$

$$a_{j+2} = \frac{(j+\alpha)(j+\alpha+1) - l(l+1)}{(j+\alpha+2)(j+\alpha+1)} a_j \quad \text{--- (17)}$$

Eq. (17) is the recurrence relation. From equations (15) & (16) $a_1 = 0$

from Eq. (17) $\longrightarrow$ $a_{odd} = 0$. Thus.

$$P_\ell(x) = \sum_{J=0,\, even}^{\infty} a_J x^{J+\alpha} \qquad (18)$$

To satisfy the eq. (17) we have two choices of α. $\alpha = 0$ or $\alpha = 1$. ~~The value of ℓ~~ $\therefore$

$\therefore$ Solution converges only for $x \to 1$ if series is finite. and series will only have finite number of terms if coefficients in eq. (17) are zero at some point

$$\frac{(J_{max}+\alpha)(J_{max}+\alpha+1) - \ell(\ell+1)}{(J_{max}+\alpha+2)(J_{max}+\alpha+1)} = 0 \qquad (19)$$

(i) Now for $\alpha = 0$.

$$J_{max}(J_{max}+1) - \ell(\ell+1) = 0$$

$$\Rightarrow J_{max} = \ell \text{; which gives.}$$

$$P_\ell(x) = \sum_{J=0,\, even}^{\ell} a_J x^J$$

or

$$\boxed{P_\ell(x) = \sum_{J=0}^{\ell/2} a_{2J} x^{2J} \quad \text{with} \quad a_{J+2} = \frac{J(J+1) - \ell(\ell+1)}{(J+2)(J+1)} a_J \text{, for } \ell = \text{even}} \qquad (20)$$

(ii) For $\alpha = 1$.

$$(J_{max}+1)(J_{max}+2) - \ell(\ell+1) = 0$$

which gives $J_{max} = \ell - 1$

$P_\ell(x)$ in this case is given by

$$P_\ell(x) = \sum_{J=0,\,even}^{\ell-1} a_J\, x^{J+1}$$

or

$$\boxed{P_\ell(x) = \sum_{J=0}^{(\ell-1)/2} a_{2J}\, x^{2J+1} \quad \text{with} \quad a_{J+2} = \frac{(J+1)(J+2) - \ell(\ell+1)}{(J+3)(J+2)}\, a_J \quad \text{for } \ell = \text{odd}} \qquad (21)$$

Writting the explicit form of $P_\ell(x)$

$P_0(x) = a_0$, $P_1(x) = a_0 x$

$P_2(x) = a_0 + a_2 x^2$ with $a_2 = -3a_0$

$P_3(x) = a_0 x + a_2 x^3$ with $a_2 = -\frac{5}{3} a_0$

or $P_0(x) = a_0$, $P_1(x) = a_0 x$

$P_2(x) = a_0(1 - 3x^2)$

$P_3(x) = a_0(x - \frac{5}{3}x^3)$

a_0 is arbitrary, conventionally we choose it such that $P_\ell(1) = 1$. Thus,

Now $P_0(x) = 1$

$P_1(x) = x$,

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Thus ~~$P_\ell(x)$ is~~

$P_\ell(x)$ is defined by Rodrigues' formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell. \qquad (22)$$

It follows the recurrence relations.

$$P_{\ell+1}(x) = \frac{2\ell+1}{\ell+1}P_\ell(x) - \frac{\ell}{(\ell+1)}P_{\ell-1}(x)\text{ ; and}$$

$$P_\ell(x) = \frac{1}{2\ell+1}\frac{d}{dx}\left[P_{\ell+1}(x) - P_{\ell-1}(x)\right]$$

The orthogonality condition is given by

$$\int_{-1}^{1} P_k(x)\,P_\ell(x)\,dx = \frac{2}{2\ell+1}\delta_{k\ell}$$

Finally, we write the general solution of Laplace equation in spherical coordinates for the case $m=0$.

$$\boxed{V(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right)P_\ell(\cos\theta)} \qquad (23)$$

where $P_\ell(x)$ are the Legendre Polynomials defined by eq. (22).